Commutative Cryptanalysis Made Practical

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Joint work with P. Felke, G. Leander, P. Neumann, L. Perrin & L. Stennes.

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Symmetric cryptography : a bit of context

Plaintext $x \in X$, ciphertext $y \in Y$, key $k \in K$. $X = \mathbb{F}_2^{n_X}, Y = \mathbb{F}_2^{n_Y}, K = \mathbb{F}_2^{n_K}$.

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A family $(E_k)_{k \in K}$, where: $\forall\, k \in K, \quad E_k : X \to Y$ is bijective.

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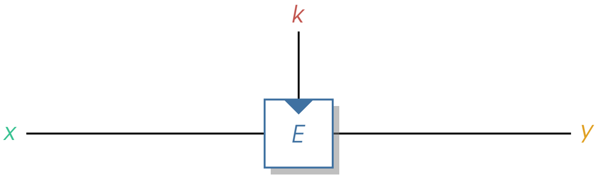
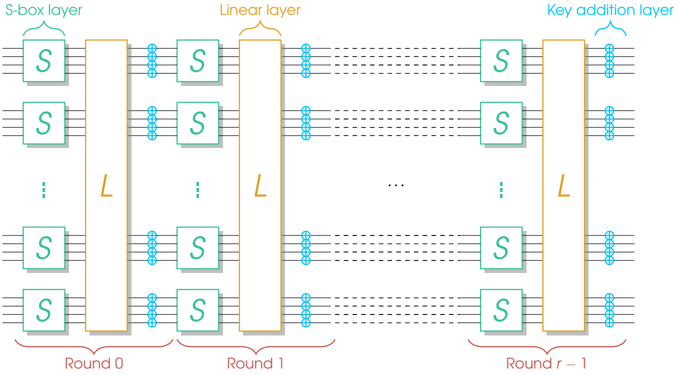


$$y = E_k(x) \iff x = (E_k)^{-1}(y)$$

$\implies$ Shared key for encryption & decryption.

Substitution Permutation Network (SPN)

- Subclass of block ciphers
- Round function, a 3-step process:
  - Local non-linear layer,
  - global linear layer,
  - and key/constant addition
- Repeat r times



Security goals

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- Modes + block cipher = confidentiality, integrity, authenticity.
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Differential distinguisher

Find α, β st. for many k, $\quad E_k(x + \alpha) = E_k(x) + \beta \quad$ has many solutions x.

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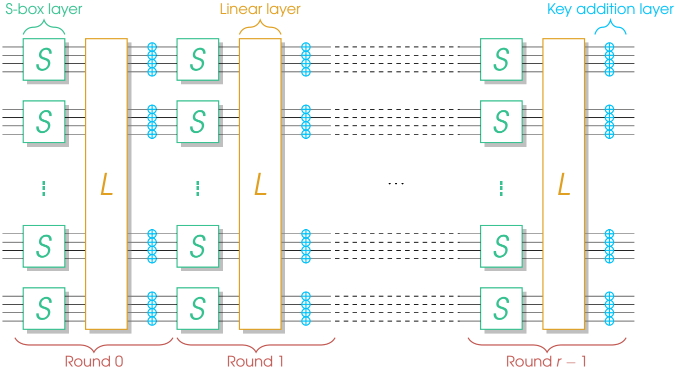
Random permutation F

$F(x+\alpha) + F(x) = \beta$ with proba 2^{-n}.

Some security arguments for an SPN

Substitution Permutation Network (SPN)

- Sbox layer, $\rightsquigarrow$ $S(x+\alpha) = S(x) + \beta$ must have few solutions for all α, β
- Linear layer, $\rightsquigarrow$ must diffuse a lot
- Key addition $\rightsquigarrow$ hard to handle...



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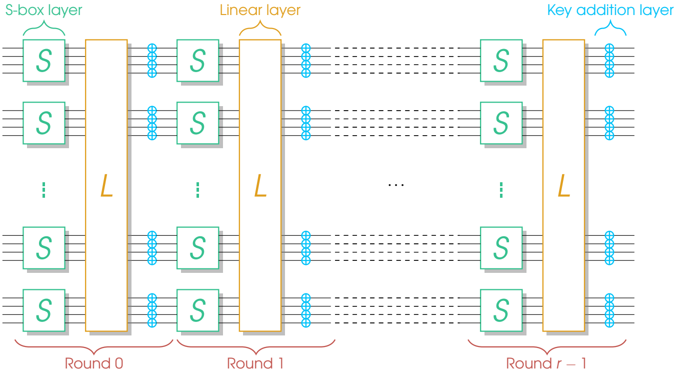
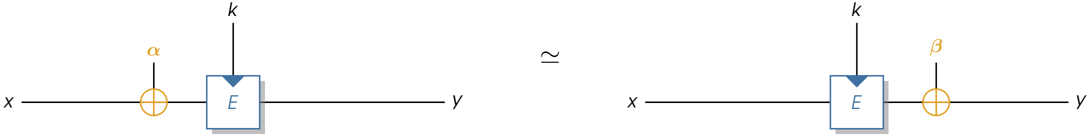


As a designer

Estimate $\mathbb{E}_{k \xleftarrow{\$} K}(\#\{x \text{ st. } E_k(x+\alpha) = E_k(x) + \beta\})$ and assume representativeness.

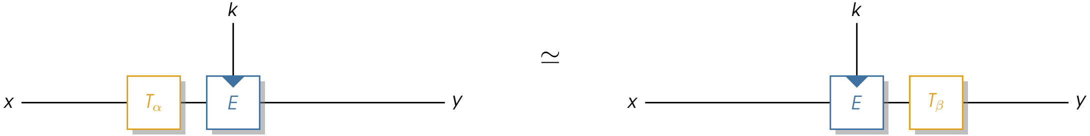
Other cryptanalysis techniques

$$E(x + \alpha) = E(x) + \beta$$



Other cryptanalysis techniques

$$E \circ T_\alpha(x) = T_\beta \circ E(x)$$



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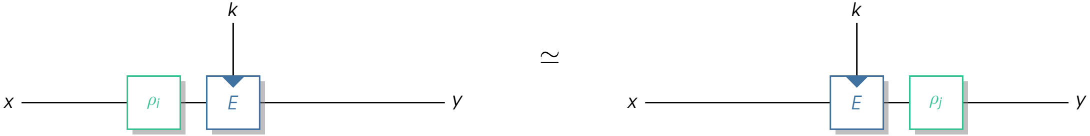
$$E \circ \rho_i(x) = \rho_j \circ E(x)$$



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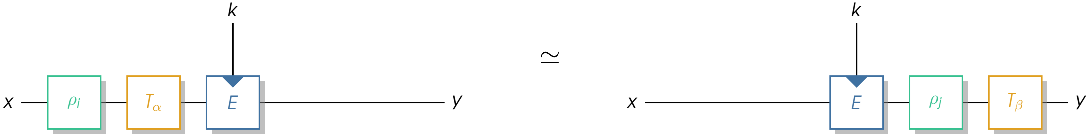
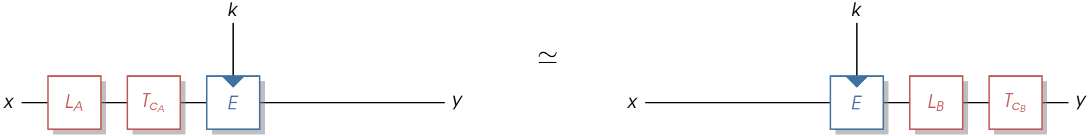


$$E \circ T_{c_A} \circ L_A(x) = T_{c_b} \circ L_B \circ E(x).$$



where $A(x) = L_A(x) + c_A, B(x) = L_B(x) + c_B$

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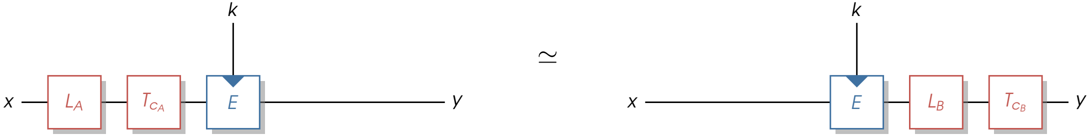

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A tempting desire of unification

Mathematically elegant, better understanding & new attacks

A 20-year-old idea [Wagner, FSE 2004]

Commutative diagram cryptanalysis: not so fruitful[1] since.

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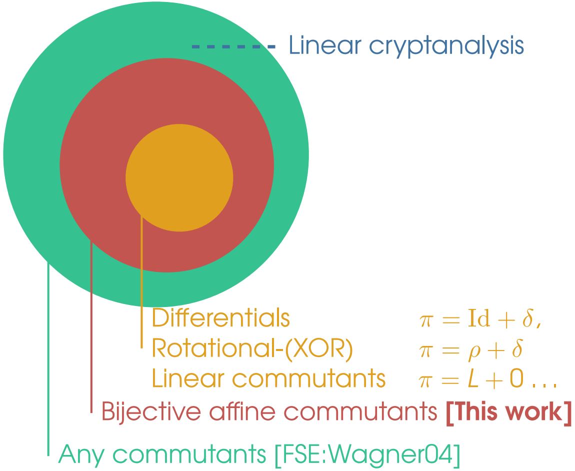
[1] to the best of our knowledge...

$$
\begin{tikzcd}
X \arrow[r, "E"] \arrow[d, "\pi_i"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "\pi_o"] \\
X' \arrow[r, "E'"] & Y'
\end{tikzcd}
$$



In this talk

Affine commutation with **probability 1**: theory + practice

A **surprising differential** interpretation

A few words about the **probabilistic case**

Goal

Find **bijective affine** A, B st. for many k: $\boxed{E_k \circ A = B \circ E_k}$ (all x are solutions)

Commutative cryptanalysis principle

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Sufficient condition for **iterated** constructions

There exist $A_0, \cdots, A_r$ st. for all i $\boxed{A_{i+1} \circ R_i = R_i \circ A_i}$.

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&= R_{r-1} \circ \cdots \circ R_1 \circ (A_1 \circ R_0) \\
&= \cdots \\
&= A_r \circ R_{r-1} \circ \cdots \circ R_0 \\
&= A_r \circ E
\end{aligned}$$

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$$\begin{tikzcd} x_0 \arrow[r, "R_0"] \arrow[d, "A_0"] & x_1 \arrow[r, dashed] \arrow[d, "A_1"] \arrow[r, phantom, shift right=6ex, "\circlearrowleft"] & x_{r-1} \arrow[r, "R_{r-1}"] \arrow[d, "A_{r-1}"] & E(x_0) \arrow[d, "A_r"] \\ z_0 \arrow[r, "R_0"'] & z_1 \arrow[r, dashed] & z_{r-1} \arrow[r, "R_{r-1}"'] & E(z_0) \end{tikzcd}$$

$\Longrightarrow$ **round-by-round** and **layer-by-layer** studies.

Layer-by-layer probability-1 trail

Simplified setting for this presentation

- Commutation only: $E \circ \mathcal{A} = \mathcal{A} \circ E$ (case $\mathcal{A} = \mathcal{B}$)
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$\mathcal{A} \circ \mathcal{S} = \mathcal{S} \circ \mathcal{A} \iff A \circ S = S \circ A \implies$ $\boxed{\text{self-affine equivalent S-box.}}$

Effective search for small m (4, 8 bits).

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Linear layer

Let $\mathcal{L} = (\mathcal{L}_{ij})$ be an invertible block matrix with m-size blocks $\mathcal{L}_{ij}$.

$\mathcal{L} \circ \mathcal{A} = \mathcal{A} \circ \mathcal{L} \iff \boxed{\mathcal{L}_{ij} \circ L_A = L_A \circ \mathcal{L}_{ij} \text{ for all } i, j \text{ and } c_{\mathcal{A}} \in \mathrm{Fix}(\mathcal{L}).}$

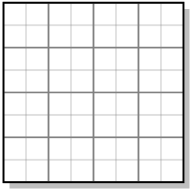
Midori [AC:BBISHA15] in a nutshell

A (not so) standard SPN

- AES-like,
- Standard wide-trail analysis,
- ...yet weak-key probability-1 (non)-linear approximations [TLS19, Bey18]
- due to (excessive) lightweightness and sparsity.

The round function

$$p = AK \circ AC \circ MC \circ PC \circ S$$



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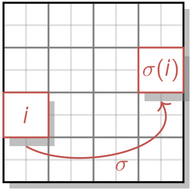

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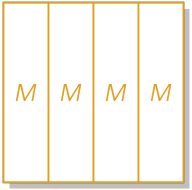

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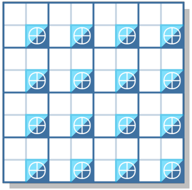

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The Midori case

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Sbox layer

There exists a single non-trivial $A^\star$ st. $A^\star \circ S = S \circ A^\star$.

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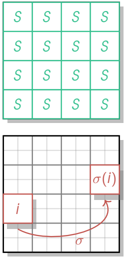
Cells permutation

Parallel mapping $\mathcal{A}$: free commutation.



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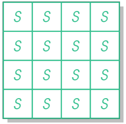

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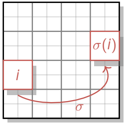

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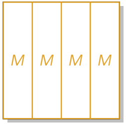

Linear layer

- $M_{ij} \circ L_A = L_A \circ M_{ij}\ \forall\, i, j$. But $M_{ij} \in \{0_4, \mathrm{Id}_4\}$.
- $c_{\mathcal{A}} \in \mathrm{Fix}(\mathcal{L})$. But $M(c,c,c,c) = (c,c,c,c)$.

$\implies$ Any $\mathcal{A}$ would work.

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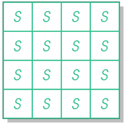

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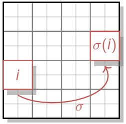

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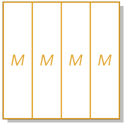

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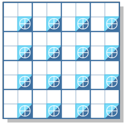

Constants

$\mathrm{Fix}(L_{A^\star}) = \langle \mathtt{0x2}, \mathtt{0x5}, \mathtt{0x8} \rangle$. $\rightsquigarrow$ Consider **variants** with modified constants.

Weak-keys: 1-bit condition per nibble $\rightsquigarrow 2^{96}$ out of 2^{128}.

The Midori case, part 2

Recap

$\mathcal{A}^\star \circ P = P \circ \mathcal{A}^\star$ for every layer P (given weak constants/keys).

$\mathcal{A}^\star \circ E_k = E_k \circ \mathcal{A}^\star$ for 1 out of 2^{32} keys k.

Recap

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$$\begin{tikzcd}
x_0 \arrow[r, "R_0"] \arrow[d, "\mathcal{A}^\star"] & x_1 \arrow[r, dashed] \arrow[d, "\mathcal{A}^\star"] & x_{r-1} \arrow[r, "R_{r-1}"] \arrow[d, "\mathcal{A}^\star"] & E(x_0) \arrow[d, "\mathcal{A}^\star"] \\
z_0 \arrow[r, "R_0"'] & z_1 \arrow[r, dashed] & z_{r-1} \arrow[r, "R_{r-1}"'] & E(z_0)
\end{tikzcd}$$

$$\mathbb{P}_{x \xleftarrow{\$} X}(\underbrace{\mathcal{A}^\star \to \mathcal{A}^\star \to \cdots \to \mathcal{A}^\star}_{r \text{ times}}) = 1, \quad \text{for any } r!$$

The Midori case, part 3

$$\begin{tikzcd}
x_0 \arrow[r, "R_0"] \arrow[d, "\Delta_0"', "\mathcal{A}^\star"] & x_1 \arrow[r, dashed] \arrow[d, "\Delta_1"', "\mathcal{A}^\star"] & x_{r-1} \arrow[r, "R_{r-1}"] \arrow[d, "\Delta_{r-1}"', "\mathcal{A}^\star"] & E(x_0) \arrow[d, "\Delta_r"', "\mathcal{A}^\star"] \\
z_0 \arrow[r, "R_0"'] & z_1 \arrow[r, dashed] & z_{r-1} \arrow[r, "R_{r-1}"'] & E(z_0)
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$$\Delta_i := x_i \oplus z_i = x_i \oplus \mathcal{A}^\star(x_i)$$

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Surprising differential interpretation

$\delta = \texttt{0xf}, \quad \Delta = \delta^{\otimes 16}, \quad \delta' = \texttt{0xa}, \quad \Delta' = \delta'^{\otimes 16}.$

$$
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$$\Delta \xrightarrow{2^{-16}} \mathcal{A}^\star \xrightarrow{1} \cdots \xrightarrow{1} \mathcal{A}^\star \xrightarrow{2^{-16}} \Delta$$

Weak-key Differential interpretation

Recap

If k is **weak**:

- $\mathbb{P}_{x \xleftarrow{\$} X}(\Delta \to \Delta') = 2^{-32}$ for any $\Delta, \Delta' \in \{\delta, \delta'\}^{16}$.
- $\mathbb{P}_{x \xleftarrow{\$} X}\left(\Delta \to \{\delta, \delta'\}^{16}\right) = 2^{-16}$ for any $\Delta \in \{\delta, \delta'\}^{16}$.
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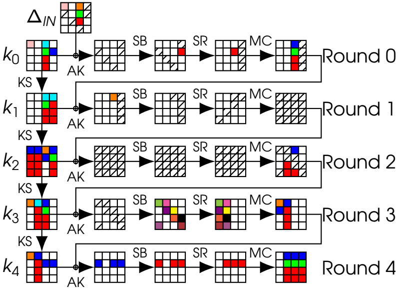
Standard case : quite low $\mathbb{P}_{k,x}$



Part of 9-round chosen-key distinguisher for AES-128.
Figure by J. Jean, extracted from Tikz for Cryptographers [Jean16].

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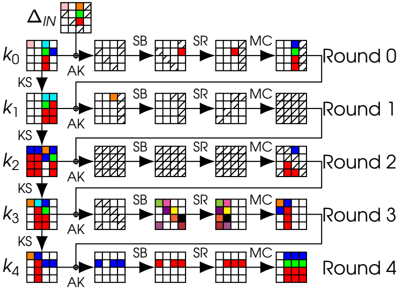

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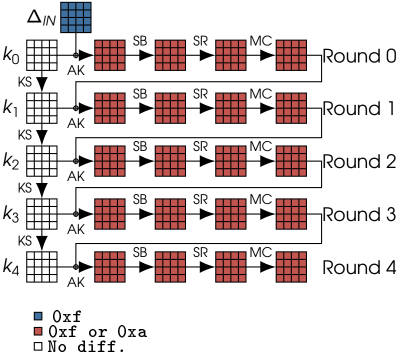
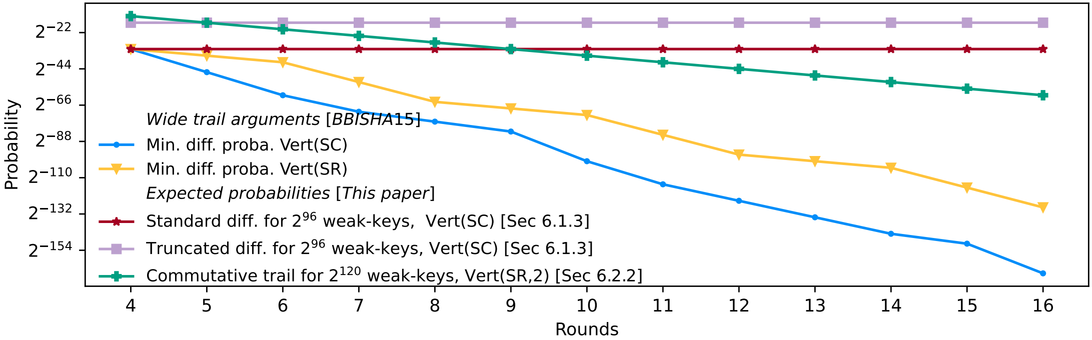
Figure by J. Jean, extracted from Tikz for Cryptographers [Jean16].

This work: high $\mathbb{P}_x$ for some k



Weak-key Differential interpretation, part 2



The designers' work

Estimate $\mathbb{E}_{k \xleftarrow{\$} K} \left(\# \{x \text{ st. } E_k(x+\alpha) = E_k(x) + \beta\}\right)$
and assume representativeness.
Blue curve.

This work

Find non-average keys with easily-distinguishable property.
Purple and red curves.

Is it that easy to detect this behavior ?

Yes !
Small demo here.

A bigger weak-key space ?

Constants

$\mathrm{Fix}(L_{A^\star}) = \langle \mathtt{0x2}, \mathtt{0x5}, \mathtt{0x8} \rangle$.

Weak-keys: 1-bit condition per nibble $\rightsquigarrow 2^{96}$ out of 2^{128}.

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$$\begin{pmatrix} A & A & A & A \\ A & A & A & A \\ A & A & A & A \\ A & A & A & A \end{pmatrix} \rightsquigarrow \widetilde{\mathcal{A}}_i = \begin{pmatrix} A^{i_0} & A^{i_4} & A^{i_8} & A^{i_c} \\ A^{i_1} & A^{i_5} & A^{i_9} & A^{i_d} \\ A^{i_2} & A^{i_6} & A^{i_a} & A^{i_e} \\ A^{i_3} & A^{i_7} & A^{i_b} & A^{i_f} \end{pmatrix}, \text{ where } A^0 = \mathrm{Id}, A^1 = A$$

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New study

- Constants : 1-bit condition if $i_j = 1$ else, 0.
- S-box: $S \circ A = A \circ S, S \circ \mathrm{Id} = \mathrm{Id} \circ S$
- Cell permutation:

$$\text{ShiftRows} \begin{pmatrix} X_0 & X_4 & X_8 & X_c \\ X_1 & X_5 & X_9 & X_d \\ X_2 & X_6 & X_a & X_e \\ X_3 & X_7 & X_b & X_f \end{pmatrix} = \begin{pmatrix} X_0 & X_4 & X_8 & X_c \\ X_5 & X_9 & X_d & X_1 \\ X_a & X_e & X_2 & X_6 \\ X_f & X_3 & X_7 & X_b \end{pmatrix}$$

A bigger weak-key space ? part 2

New pattern

$$\begin{pmatrix} A & \mathrm{Id} & A & \mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} & \mathrm{Id} & \mathrm{Id} \\ A & \mathrm{Id} & A & \mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} & \mathrm{Id} & \mathrm{Id} \end{pmatrix}$$

Commutes with S-box layer, cells perm. and weak-constants/weak-key addition.

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What about M ?

$$M := \begin{pmatrix} 0 & \mathrm{Id} & \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & 0 & \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} & 0 & \mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} & \mathrm{Id} & 0 \end{pmatrix}$$

We indeed have $M_{ij} \in \{0, \mathrm{Id}\}$ and $M(c, c, c, c) = (c, c, c, c)$.

A bigger weak-key space ? part 3

$$M := \begin{pmatrix} 0 & \mathrm{Id} & \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & 0 & \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} & 0 & \mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} & \mathrm{Id} & 0 \end{pmatrix}$$

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$$M \begin{pmatrix} Ax_0 \\ x_1 \\ Ax_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + Ax_2 + x_3 \\ Ax_0 + Ax_2 + x_3 \\ Ax_0 + x_1 + x_3 \\ Ax_0 + x_1 + Ax_2 \end{pmatrix} = \begin{pmatrix} x_1 + L_A x_2 + x_3 + c_A \\ L_A x_0 + L_A x_2 + x_3 \\ L_A x_0 + x_1 + x_3 + c_A \\ L_A x_0 + x_1 + L_A x_2 \end{pmatrix} \tag{1}$$

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A bigger weak-key space ? part 3

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$$(1) = (2) \iff \begin{pmatrix} L_A x_1 + x_1 + L_A x_3 + x_3 \\ L_A x_0 + x_0 + L_A x_2 + x_2 \\ L_A x_1 + x_1 + L_A x_3 + x_3 \\ L_A x_0 + x_0 + L_A x_2 + x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{matrix} (L_A + \mathrm{Id})(x_0 + x_2) = 0 \\ (L_A + \mathrm{Id})(x_1 + x_3) = 0 \end{matrix}$$

A bigger weak-key space ? final

Recap

$A \times \mathrm{Id} \times A \times \mathrm{Id} \circ M(x_0, x_1, x_2, x_3) = M\,(Ax_0, x_1, Ax_2, x_3)$ if and only if $x_0 + x_2 \in \ker(L_A + \mathrm{Id})$ and $x_1 + x_3 \in \ker(L_A + \mathrm{Id})$

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Fact

$\dim(\ker(L_A + \mathrm{Id})) = 2.$

First probabilistic commutation, first trade-off

$$\mathbb{P}_{x \xleftarrow{\$} X}\left(\mathcal{A} \circ \mathcal{M}(x) = \mathcal{M} \circ \mathcal{A}(x)\right) = \frac{2^2}{2^4} \times \frac{2^2}{2^4} = 2^{-4}.$$

For $2^{128-2\times4} = 2^{120}$ weak keys, $\mathbb{P}_{x \xleftarrow{\$} X}\left(R \circ \mathcal{M}(x) = \mathcal{M} \circ R(x)\right) = 2^{-4}$.

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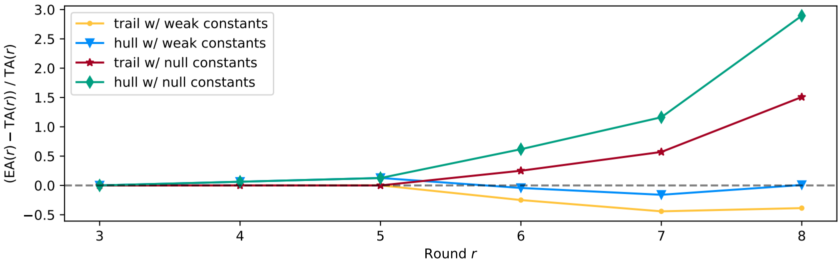
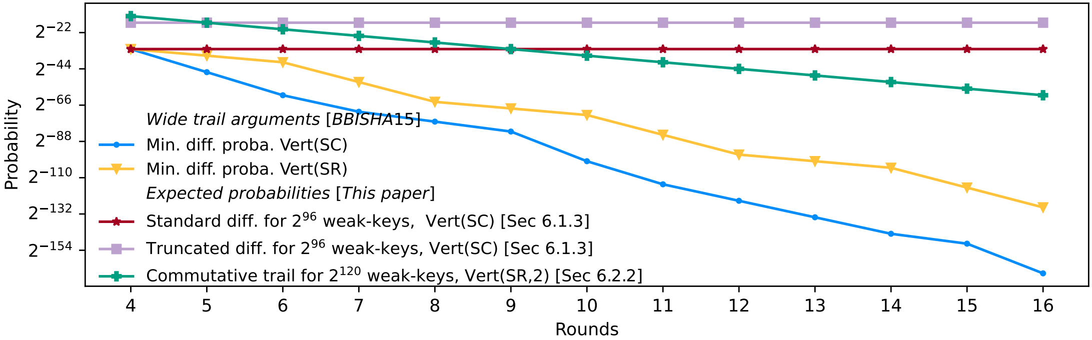


A few words about probabilistic commutation



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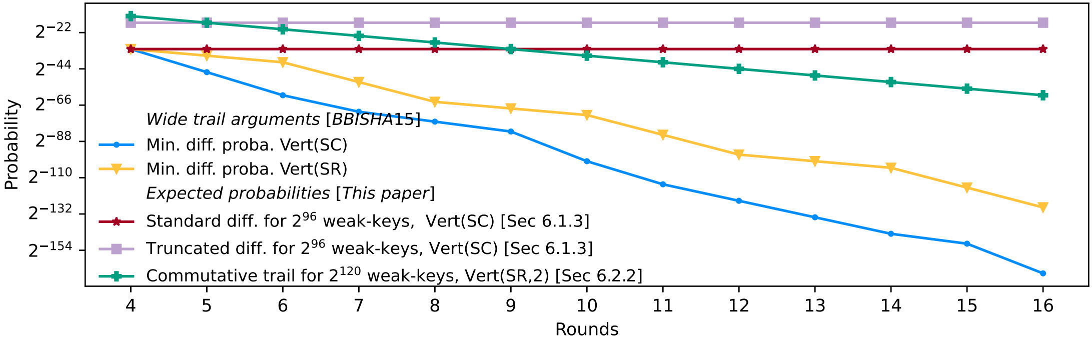

Probabilistic commutation with different layers

Let $p \in [0, 1]$.

- $A \circ T_k \overset{p}{=} T_k \circ B$: well-understood.
- $A \circ L \overset{p}{=} L \circ B$: manageable for parallel mappings.
- $A \circ S \overset{p}{=} S \circ B$: 4-bit mappings can be listed exhaustively.

Conclusion

In practice

- Trade-offs: number-of-weak-keys VS probability-of-success.
- Independence of rounds must be supposed ...but often too optimistic.

Further studies

- Algorithm for probabilistic affine-equivalence.
- Study the dependencies.
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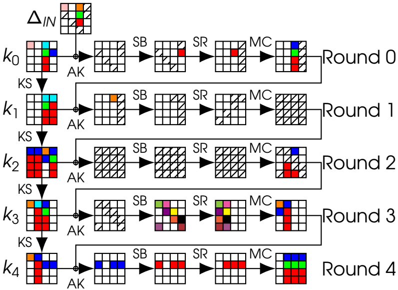

This work: high $\mathbb{P}_x$ for some weak k