Differential cryptanalysis of conjugate ciphers

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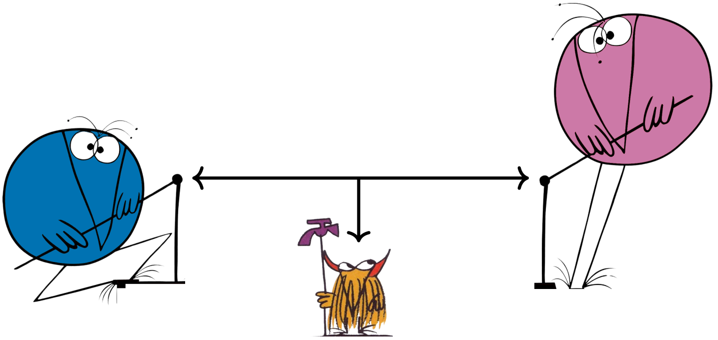
based on joint works with C. Beierle, P. Felke, G. Leander, P. Neumann, L. Perrin & L. Stennes

UCLouvain

Almasty seminar, January 17th, 2025

Assumption

Common secret 🔑 shared beforehand.



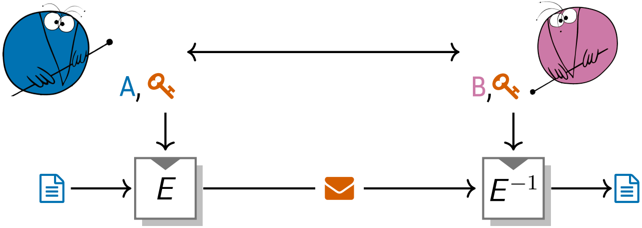
Goal

Ensure ***confidentiality*** and/or authenticity and/or integrity

Symmetric encryption

Goal

Ensure confidentiality



Constraints

- Secure
- Easily implemented
- Arbitrary-long messages

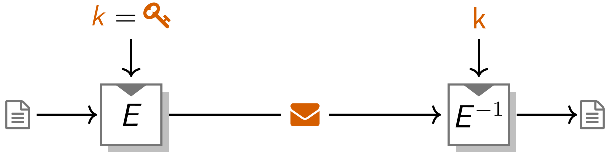
Definition (Primitive)

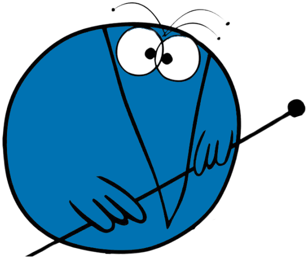
Low-level algorithm for *very specific* tasks

Example (Block cipher)

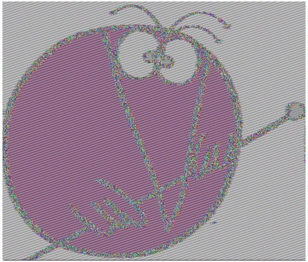
Encrypts *fixed-size* messages

$\rightsquigarrow$ A block cipher $\mathcal{E}$ is a family of bijections $\mathcal{E} = \left(E_k : \mathbb{F}_2^n \xrightarrow{\sim} \mathbb{F}_2^n\right)_{k \in \mathbb{F}_2^\kappa}$.





Each pixel is encrypted independently



Definition (Mode of operation)

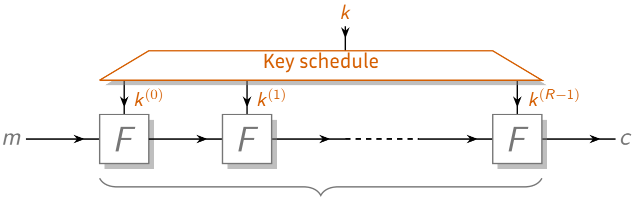
High-level algorithm *based on primitives* to provide *e.g.* confidentiality

Building a block cipher

Recap (Block cipher)

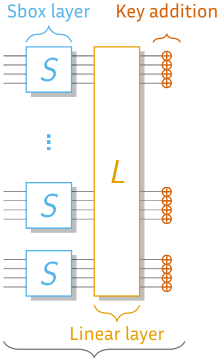
A family of bijections $\mathcal{E} = \left(E_k : \mathbb{F}_2^n \xrightarrow{\sim} \mathbb{F}_2^n\right)_{k \in \mathbb{F}_2^\kappa}$. Should be *efficient* and *secure*.

Iterated construction



$$E_k = F_{k^{(R-1)}} \circ \cdots \circ F_{k^{(1)}} \circ F_{k^{(0)}}$$

Substitution Permutation Network



$$F_{k^{(i)}} = T_{k^{(i)}} \circ L \circ S$$

Recap (Block cipher)

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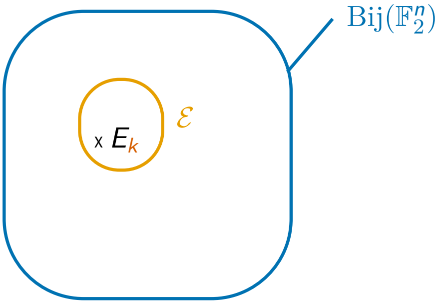

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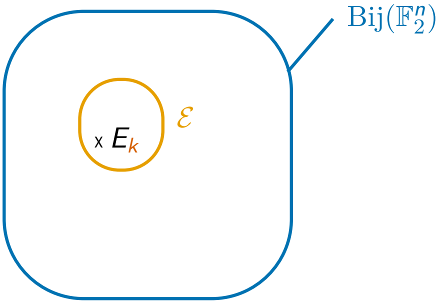

Definition (Indistinguishability)

$[\, E \xleftarrow{\$} \mathcal{E} \,]$ ***indistinguishable*** from $[\, F \xleftarrow{\$} \mathrm{Bij}(\mathbb{F}_2^n) \,]$.

Outline

II - Differential cryptanalysis

Recap

$\mathcal{E} = \left(E_k : \mathbb{F}_2^n \xrightarrow{\sim} \mathbb{F}_2^n\right)_{k \in \mathbb{F}_2^\kappa}$. $\qquad$ $[\, E \xleftarrow{\$} \mathcal{E} \,]$ or $[\, F \xleftarrow{\$} \mathrm{Bij}(\mathbb{F}_2^n) \,]$?

The difference Δ^{out} between two ciphertexts ***should be uniformly distributed***, even when the difference Δ^{in} between plaintexts is chosen.

$x \updownarrow \Delta^{\mathrm{in}}\, y \qquad\qquad E_k(x) \updownarrow \Delta^{\mathrm{out}}\, E_k(y)$

Differential distinguisher

Recap

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For a random bijection F

$F(x + \Delta^{\mathrm{in}}) + F(x) = \Delta^{\mathrm{out}}$ has 1 solution x on average.

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Differential distinguisher [BihSha91]

$\Delta^{\text{in}} \neq 0, \Delta^{\text{out}}$ s.t for many k, $E_k(x + \Delta^{\text{in}}) + E_k(x) = \Delta^{\text{out}}$ has many solutions x.

$$
\begin{array}{ccccccc}
x^{(0)} & \xrightarrow{F_{k^{(0)}}} & x^{(1)} & \dashrightarrow & x^{(R-1)} & \xrightarrow{F_{k^{(R-1)}}} & E_k(x^{(0)}) \\
\updownarrow \Delta^{\text{in}} & & \updownarrow \Delta^{(1)} & & \updownarrow \Delta^{(R-1)} & & \updownarrow \Delta^{\text{out}} \\
y^{(0)} & \xrightarrow[F_{k^{(0)}}]{} & y^{(1)} & \dashrightarrow & y^{(R-1)} & \xrightarrow[F_{k^{(R-1)}}]{} & E_k(y^{(0)})
\end{array}
$$

$F_{k^{(i)}} = F \circ T_{k^{(i)}}$ for $i \geq 0$.

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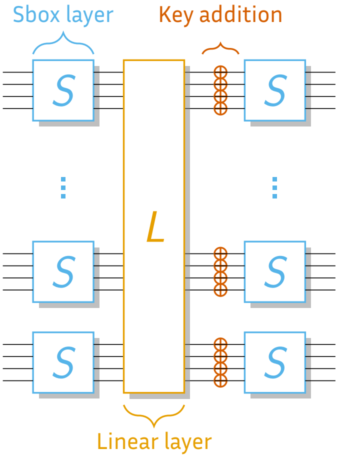
$F_{k^{(i)}} = F \circ T_{k^{(i)}}$ for $i \geq 0$.

On average over all key sequences [LaiMasMur91]

$$\mathbb{E}\left[\Delta^{(0)} \xrightarrow{\mathcal{E}} \Delta^{(r)}\right] \geq \mathbb{E}\left[\Delta^{(0)} \xrightarrow{F} \Delta^{(1)} \rightarrow \cdots \xrightarrow{F} \Delta^{(R)}\right] = \prod_{i=0}^{R-1} \mathbb{P}\left[\Delta^{(i)} \xrightarrow{F} \Delta^{(i+1)}\right]$$

Resisting differential cryptanalysis



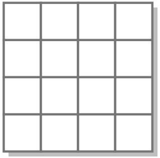
As a designer [DaeRij00]

- Low ***differential uniformity***: [Nyberg94]

$$\delta(S) = \max_{\Delta^{\text{in}} \neq 0, \Delta^{\text{out}}} \left| \left\{ x, S(x + \Delta^{\text{in}}) + S(x) = \Delta^{\text{out}} \right\} \right|$$

- Minimum number of active Sboxes determined by L

Advanced Encryption Standard (AES)



AES [DaeRij00]

- 4×4 matrix of bytes = 128-bit state

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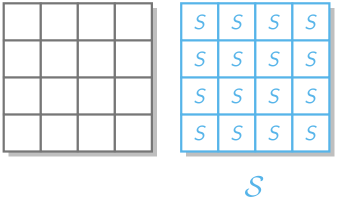

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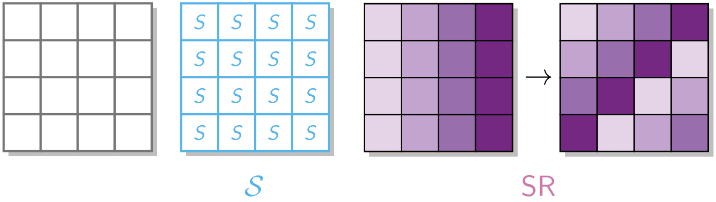

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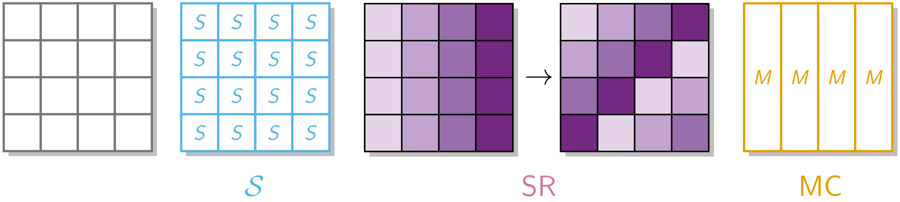

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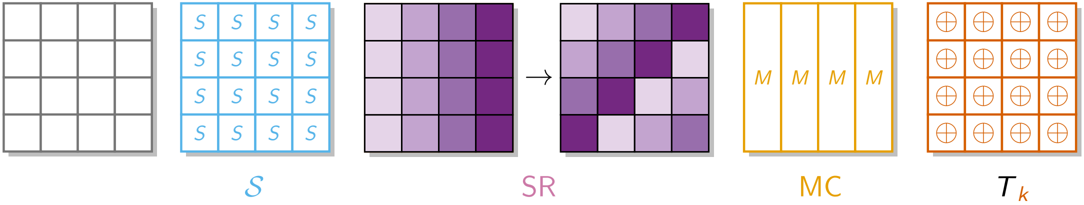
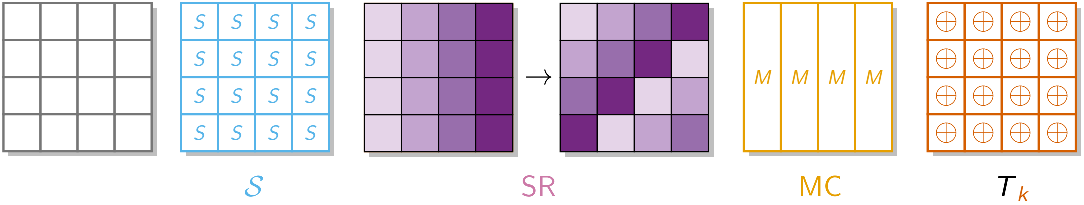


AES

[DaeRij00]

- 4×4 matrix of bytes = 128-bit state
- $F_{k(i)} = T_{k(i)} \circ \mathrm{MC} \circ \mathrm{SR} \circ \mathcal{S}$.
- Repeat 10 times.

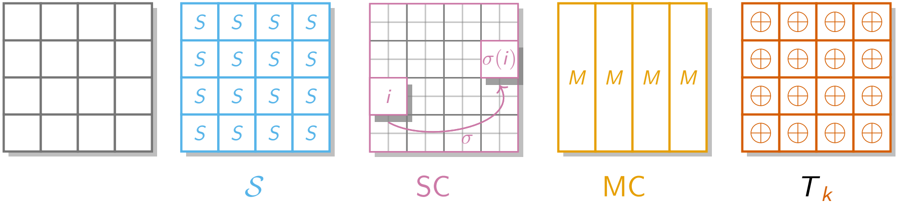
Advanced Encryption Standard (AES)



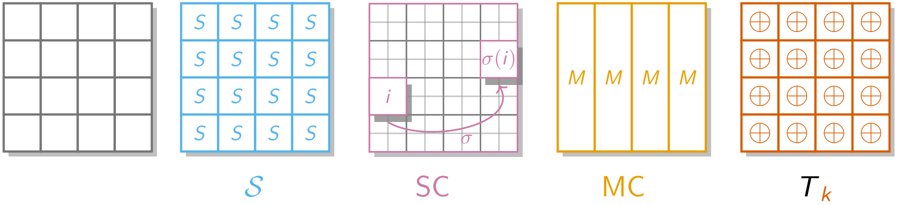
AES [DaeRij00]

- 4×4 matrix of bytes = 128-bit state
- $F_{k^{(i)}} = T_{k^{(i)}} \circ \mathsf{MC} \circ \mathsf{SR} \circ \mathcal{S}$.
- Repeat 10 times.
- $\delta(S) = 4$.
- Structured linear layer $\mathsf{MC} \circ \mathsf{SR}$: $\implies \mathbb{E}\left[\Delta^{(0)} \xrightarrow{F^{(0)}} \Delta^{(1)} \to \cdots \xrightarrow{F^{(3)}} \Delta^{(3)}\right] \leq 2^{-150}$.

Midori



Midori



Midori [BBISHAR15]

- 4×4 matrix of ***nibbles*** = 64-bit state
- $F_{k^{(i)}} = T_{k^{(i)}} \circ \mathrm{MC} \circ \mathrm{SC} \circ \mathcal{S}$.
- Repeat 16 times.
- $\delta(S) = 4$.
- $\mathbb{E}\left[\Delta^{(0)} \xrightarrow{F^{(0)}} \Delta^{(1)} \rightarrow \dots \xrightarrow{F^{(6)}} \Delta^{(7)}\right] \leq 2^{-70}$.

III - Differential cryptanalysis of conjugate ciphers

Changing our point of view

Chosen plaintext access = freedom of study

1) Encrypt $H(x)$ $\rightsquigarrow E_k \circ H(x)$
2) Apply G $\rightsquigarrow G \circ E_k \circ H(x)$
3) Study $G \circ E_k \circ H$

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Conjugation

The conjugate of F relative to G is the function $G \circ F \circ G^{-1}$ denoted by F^G.

F^G is the ***same function*** as F, ***up to a change of variables***.

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Proof left as exercice. □ $(G^{-1} \circ G = \mathrm{Id})$

Changing our point of view

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Proof left as exercice. □ $(G^{-1} \circ G = \mathrm{Id})$

Is it simpler to attack E_k^G than E_k ?

Linear VS non-linear change of variables

Recap

$F^G := G \circ F \circ G^{-1}$

$$E_k^G = F_{k(R-1)}^G \circ \ldots \circ F_{k(1)}^G \circ F_{k(0)}^G$$

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Definition/Proposition (Affine equivalence)

Def: $F_1 \sim_{\mathrm{aff}} F_2$ if $\exists\, A, B$ bijective affine s.t. $A \circ F_1 \circ B = F_2$.

Prop: If $F_1 \sim_{\mathrm{aff}} F_2$, then $\delta(F_1) = \delta(F_2)$ and $\mathcal{L}(F_1) = \mathcal{L}(F_2)$

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Corollary

- If G linear, $\delta(F) = \delta(F^G)$ and $\mathcal{L}(F) = \mathcal{L}(F^G)$

  $\implies$ Fine-grained arguments are needed.

- If G non-linear ?

  $\implies$ Linear attack cf. [BeiCanLea18]

  $\implies$ Differential attack cf. [BFLNPS23,BBFLNPS24]

Non-linear change of variables (1/3)

$$F_{k^{(i)}} = T_{k^{(i)}} \circ \mathrm{MC} \circ \mathrm{SC} \circ \mathcal{S} \quad \rightsquigarrow \quad F^{G}_{k^{(i)}} = T^{G}_{k^{(i)}} \circ \mathrm{MC}^{G} \circ \mathrm{SC}^{G} \circ \mathcal{S}^{G}$$

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Main problem

If F is *linear*, F^G is a priori not.

$\implies$ T^G_k *non-linear dependency* in the key bits.

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A possible solution

General case For all Δ and all k: $\mathbb{P}\left[\Delta \xrightarrow{T_k} \Delta\right] = 1$

Conjugated case For *some* Δ and *some* k: $\mathbb{P}\left[\Delta \xrightarrow{T^G_k} \Delta\right] = 1$

$\implies$ ***Weak-key attacks***!

Non-linear change of variables (2/3)

Recap

Conjugated case For *some* Δ and *some* k: $\mathbb{P}\left[\Delta \xrightarrow{T_k^G} \Delta\right] = 1$

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Weak-key space

$$W(\Delta) = \left\{k,\ \mathbb{P}\left[\Delta \xrightarrow{T_k^G} \Delta\right] = 1\right\}$$

$$\mathbb{P}\left[\Delta \xrightarrow{T_k^G} \Delta\right] = 1 \iff \forall\, x,\ T_k^G(x) + T_k^G(x + \Delta) = \Delta$$

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Definition (Derivative)

The function $D_\Delta F : x \mapsto F(x) + F(x + \Delta)$ is the ***derivative*** of F along the direction Δ.

Non-linear change of variables (2/3)

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$$\mathbb{P}\left[\Delta \xrightarrow{T_k^G} \Delta\right] = 1 \iff D_\Delta T_k^G \text{ is constant}$$

Non-linear change of variables (3/3)

Intuition

T_k^G with constant derivatives $\rightsquigarrow$ $T_k^G = G \circ T_k \circ G^{-1}$ somehow close to be linear.

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Our explored space

$\mathcal{G}$ Sbox layer based on $G : \mathbb{F}_2^4 \to \mathbb{F}_2^4$ with

$$G\,(x_0,\; x_1,\; x_2,\; x_3) = (x_0 + g(x_1, x_2, x_3),\; x_1,\; x_2,\; x_3)$$

$(G = G^{-1})$

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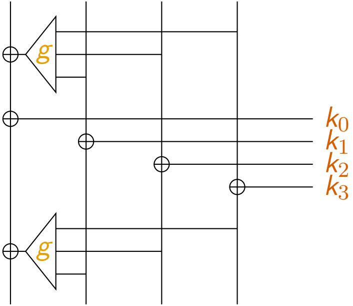

$$T_k^G(x_0,\ x_1,\ x_2,\ x_3) = \begin{pmatrix} x_0 + k_0 + D_{\tilde{k}}g(x_1, x_2, x_3) \\ x_1 + k_1 \\ x_2 + k_2 \\ x_3 + k_3 \end{pmatrix}$$

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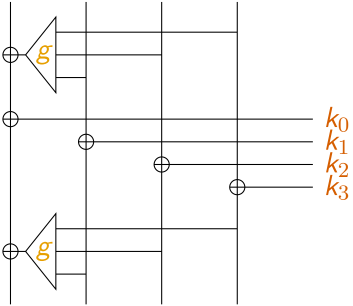

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g quadratic $\Longrightarrow$ T_k^G linear $\Longrightarrow$ constant derivatives $D_\Delta T_k^G$

Sbox

By computer search, there exist G and Δ s.t $\mathbb{P}\left[\Delta \xrightarrow{S^G} \Delta\right] = 1$ $\mathbb{P}\left[\nabla \xrightarrow{\mathcal{S}^{\mathcal{G}}} \nabla\right] = 1.$

$\nabla = (\Delta, \ldots, \Delta)$.

The case of Midori

Sbox

By computer search, there exist G and Δ s.t $\mathbb{P}\left[\Delta \xrightarrow{S^G} \Delta\right] = 1$ $\mathbb{P}\left[\nabla \xrightarrow{S^{\mathcal{G}}} \nabla\right] = 1.$

$\nabla = (\Delta, \dots, \Delta).$

Linear layer

$$M = \begin{pmatrix} 0 & \mathrm{Id} & \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & 0 & \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} & 0 & \mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} & \mathrm{Id} & 0 \end{pmatrix}$$

$$\mathbb{P}\left[\nabla \xrightarrow{MC^{\mathcal{G}}} \nabla\right] = 1$$

The case of Midori

Sbox

By computer search, there exist G and Δ s.t $\mathbb{P}\left[\Delta \xrightarrow{S^G} \Delta\right] = 1$ $\mathbb{P}\left[\nabla \xrightarrow{S^{\mathcal{G}}} \nabla\right] = 1.$

$\nabla = (\Delta, \ldots, \Delta)$.

Linear layer

$$M = \begin{pmatrix} 0 & \mathrm{Id} & \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & 0 & \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} & 0 & \mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} & \mathrm{Id} & 0 \end{pmatrix}$$

$$\mathbb{P}\left[\nabla \xrightarrow{\mathsf{MC}^{\mathcal{G}}} \nabla\right] = 1$$

Probability-1 distinguisher for infinitely many rounds⋆

$$\mathbb{P}\left[\nabla \xrightarrow{S^{\mathcal{G}}} \nabla \xrightarrow{(MC\circ\mathsf{SC})^{\mathcal{G}}} \nabla \xrightarrow{T^{\mathcal{G}}_{k(0)}} \nabla \xrightarrow{S^{\mathcal{G}}} \nabla \xrightarrow{(MC\circ\mathsf{SC})^{\mathcal{G}}} \nabla \xrightarrow{T^{\mathcal{G}}_{k(1)}} \nabla \xrightarrow{S^{\mathcal{G}}} \nabla \xrightarrow{(MC\circ\mathsf{SC})^{\mathcal{G}}} \nabla \xrightarrow{T^{\mathcal{G}}_{k(0)}} \cdots\right] = 1$$

⋆ If the two round keys are weak. $\frac{|W(\nabla)|}{2^{64}} = 2^{-16} \implies 2^{96}$ weak-keys for variants of Midori

Equivalent points of view

$$\mathbb{P}\left[\Delta^{\text{in}} \xrightarrow{F^{G}} \Delta^{\text{out}}\right] = 1 \quad \Longleftrightarrow \quad \forall\, x, F^{G}(x + \Delta^{\text{in}}) + F^{G}(x) = \Delta^{\text{out}}$$

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- "Commutation" $F \circ A = B \circ F$ [BFLNPS23]

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- "Commutation" $F \circ A = B \circ F$ [BFLNPS23]
- Self-equivalence $B^{-1} \circ F \circ A = F$ [BFLNPS23]
- Differential eq. for ***another group law*** $F \circ \left(G^{-1} \circ T_{\Delta^{\text{in}}} \circ G\right) = \left(G^{-1} \circ T_{\Delta^{\text{out}}} \circ G\right) \circ F$
  $G^{-1} T_{\Delta} G$ is an addition, up to a change of variables. [CivBloSal19, CalCivInv24]

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The case of Midori

- $A = B \implies$ "commutation" makes sense
- A and B are affine $\implies$ Self-equivalence makes sense

Benefits from each point of view

$$\mathbb{P}\left[\Delta^{\text{in}} \xrightarrow{F^G} \Delta^{\text{out}}\right] = 1 \iff F \circ \left(G^{-1} \circ T_{\Delta^{\text{in}}} \circ G\right) = \left(G^{-1} \circ T_{\Delta^{\text{out}}} \circ G\right) \circ F$$
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Self affine-equivalence for the Sbox

Efficient search for affine bijections A, B s.t. $B^{-1} \circ F \circ A = F$ [BDBP03][Dinur18]

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Commutation for linear layer

For Midori, A affine and $A = B$.

$$\begin{pmatrix} 0 & \mathrm{Id} & \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & 0 & \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} & 0 & \mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} & \mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{pmatrix} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{pmatrix} \begin{pmatrix} 0 & \mathrm{Id} & \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & 0 & \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} & 0 & \mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} & \mathrm{Id} & 0 \end{pmatrix}$$

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Alternative group law for key addition layer

Bounds on the dimension of $W(\Delta)$. [CivBloSal19]

Take away

Differential cryptanalysis of conjugates ***makes sense***

Theorem (Many fruitful points of view)

Commutative $\supset$ Affine commutative $\approx$ Differential for conjugates = Differential w.r.t $(\mathbb{F}_2^n, \diamond)$

Open questions

- Efficient ways of finding "good" G?
- ***Probabilistic*** cryptanalysis
- Associated ***security criteria*** ?

IV - Relationship with standard differential cryptanalysis

Recap (Commutative interpretation for "almost"-Midori)

Under weak-key hypothesis, there exists an affine bijective mapping $\mathcal{A}$ such that:

$$\mathcal{A} \circ F = F \circ \mathcal{A} \quad \text{for every layer } F.$$

$$\begin{array}{ccccccc}
x^{(0)} & \xrightarrow{F_{k^{(0)}}} & x^{(1)} & \dashrightarrow & x^{(R-1)} & \xrightarrow{F_{k^{(0)}}} & E_k(x^{(0)}) \\
\downarrow \mathcal{A} & & \downarrow \mathcal{A} & & \downarrow \mathcal{A} & & \downarrow \mathcal{A} \\
y^{(0)} & \xrightarrow[F_{k^{(0)}}]{} & y^{(1)} & \dashrightarrow & y^{(R-1)} & \xrightarrow[F_{k^{(0)}}]{} & E_k(y^{(0)})
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From commutative cryptanalysis back to differential cryptanalysis

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Differential cryptanalysis

Commutative cryptanalysis restricted to $\mathcal{A}(x) = \mathrm{Id}(x) + \Delta$

$$\begin{array}{ccccccc}
x^{(0)} & \xrightarrow{F_{k^{(0)}}} & x^{(1)} & \dashrightarrow & x^{(R-1)} & \xrightarrow{F_{k^{(R-1)}}} & E_k(x^{(0)}) \\
\updownarrow \Delta^{\mathrm{in}} & & \updownarrow \Delta^{(1)} & & \updownarrow \Delta^{(R-1)} & & \updownarrow \Delta^{\mathrm{out}} \\
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Differential interpretation of a commutative distinguisher

$$
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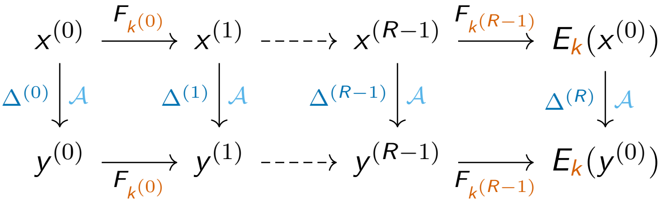
$$\Delta^{(i)} := x^{(i)} \oplus y^{(i)} = x^{(i)} \oplus \mathcal{A}(x^{(i)})$$

Differential interpretation of a commutative distinguisher



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Observation

Let $C : x \mapsto x \oplus A(x)$. Then $C(\mathbb{F}_2^4) = \{\delta, \delta'\}$ where $\delta \neq \delta'$.

$$\forall\, \Delta \in \{\delta, \delta'\}^{16},\ \mathbb{P}_{x \xleftarrow{\$} \mathbb{F}_2^{64}} (x + \mathcal{A}(x) = \Delta) = 2^{-16}$$

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Surprising differential interpretation

A differential pair $(x, x + \Delta)$ coincides with a commutative pair $(x, \mathcal{A}(x))$ with proba 2^{-16}

$$\Delta \xrightarrow{2^{-16}} \mathcal{A} \xrightarrow{1} \cdots \xrightarrow{1} \mathcal{A} \xrightarrow{2^{-16}} \Delta$$

Weak-key differential interpretation

Recap

Under weak-key hypothesis:

- $\mathbb{P}_{x \xleftarrow{\$} X}\left(\Delta \to \{\delta, \delta'\}^{16}\right) \geq 2^{-16}$ for any $\Delta \in \{\delta, \delta'\}^{16}$.
- If output differences are uniformly distributed, then: $\mathbb{P}_{x \xleftarrow{\$} X}\left(\Delta \to \Delta'\right) \approx 2^{-32}$ for any $\Delta, \Delta' \in \{\delta, \delta'\}^{16}$
- Holds for *infinitely many rounds* !

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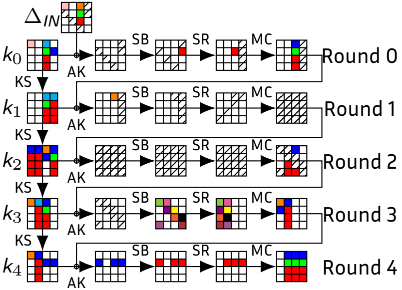
Standard case : *quite low* $\mathbb{P}_{k,x}$



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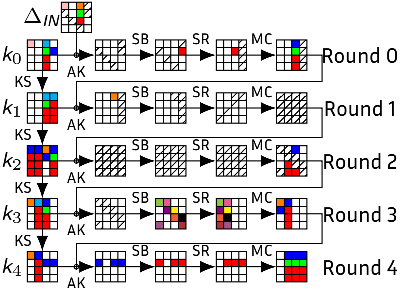
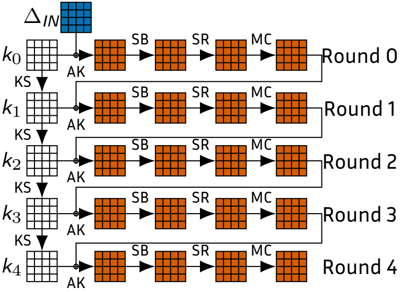
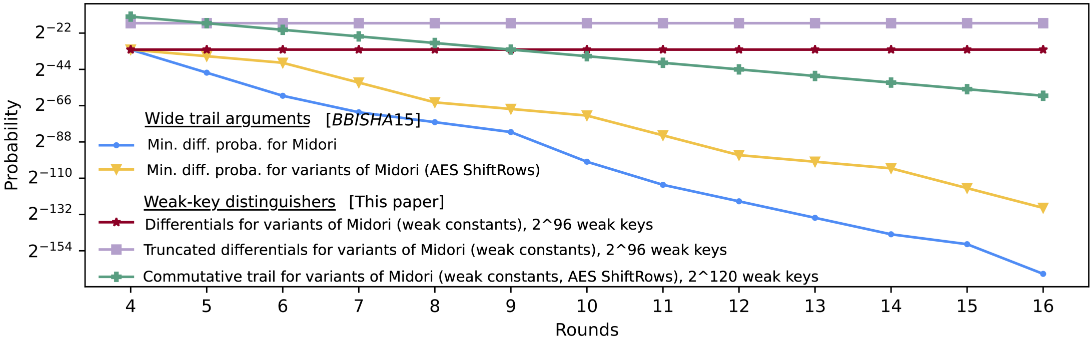


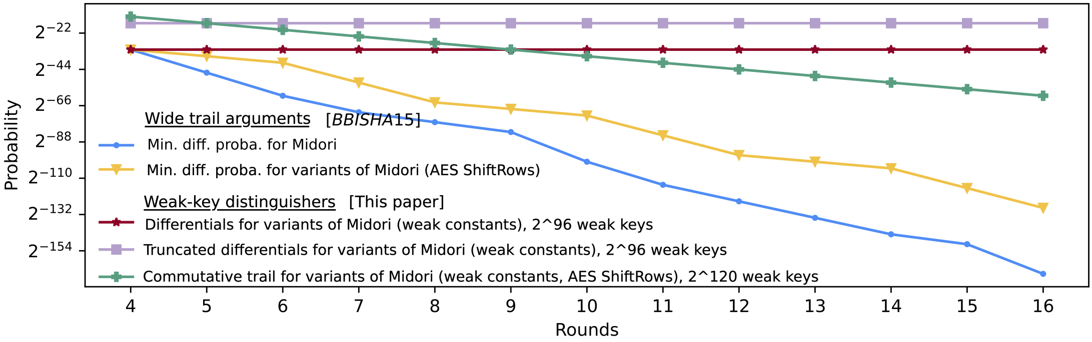
This work: *high* $\mathbb{P}_x$ for *some* k



Weak-key Differential interpretation, part 2



Weak-key Differential interpretation, part 2



Caution

- *Same observations* for the CAESAR candidate SCREAM.
- *Same idea* can be used to *hide* probability-1 differential trails. [C:BFLNS23]

Good news

Probability-1 commutative trails can be *automatically* detected !

Take away

Conjugates of ciphers do *play a role in cryptanalysis*

Differential cryptanalysis

- Efficient ways of finding "good" G?
- *Probabilistic* cryptanalysis
- Associated *security criteria* ?

Systematization of change of variables in cryptanalysis?

- Linear using non-linear G [BeiCanLea18]
- Differential using non-linear G [BFLNPS23,BBFLNPS24]
- Integral using linear G [DerFou20,DerFouLam20]

Change of variables in design?

Classification of known optimal functions w.r.t differential cryptanalysis [BCanPer24]