



Boolean Modeling and Analysis of Learning With Rounding

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Outline

- 1 Introduction: motivation and setting
- 2 A symmetric point of view
- 3 Results
- 4 Some intuition
- 5 A taste of the algorithms

Hard learning problems

Learning With Error (LWE), Learning With Rounding (LWR), Learning Parity with Noise (LPN)

and their ring/module variants.

Central importance in [post-quantum cryptography](#)

- Encryption, Key encapsulation mechanisms: CRYSTALS-Kyber, Saber
- Signatures: CRYSTALS-Dilithium, BLISS

and in [symmetric cryptography](#):

- Essentially to build (key homomorphic) PRFs for a variety of applications.
- E.g. distributed PRFs, proxy re-encryption, updatable encryption (Boneh et al., 2013)

Learning With Errors

In a nutshell: solving a **noisy linear system** over a ring.

Search Learning With Errors (Regev 05)

Parameters: $q \in \mathbb{N}$, $n \in \mathbb{N}^*$, **small (Gaussian) distribution** χ over \mathbb{Z}_q , **secret** $\mathbf{x} \xleftarrow{\$} \mathbb{Z}_q^n$

Given samples from the distribution

$$\mathcal{D}^{\text{LWE}} = \{ (\mathbf{a}, \langle \mathbf{a}, \mathbf{x} \rangle + e), \mathbf{a} \xleftarrow{\$} \mathbb{Z}_q^n, e \leftarrow \chi \}$$

Find \mathbf{x} .

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- Security level is determined by n , q , and **standard deviation** σ of χ .
- Drawback: LWE cannot be used to build **deterministic primitives** such as PRFs.

Learning with Rounding

'A way of partially *derandomizing*' the LWE problem, i.e. generating errors efficiently and *deterministically*'.

Banerjee, Peikert, Rosen, EC' 2012.

Search Learning With Rounding

Parameters: $q \in \mathbb{N}$, $p, n \in \mathbb{N}^*$, $p < q$, rounding function $\lfloor \cdot \rfloor_p : \mathbb{Z}_q \rightarrow \mathbb{Z}_p$, secret $\mathbf{x} \xleftarrow{\$} \mathbb{Z}_q^n$

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$$\mathcal{D}^{\text{LWR}} = \{ (\mathbf{a}, \lfloor \langle \mathbf{a}, \mathbf{x} \rangle \rfloor_p), \mathbf{a} \xleftarrow{\$} \mathbb{Z}_q^n \}$$

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Power-of-two LWR

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In this case:

- rounding function $\lfloor \cdot \rfloor_p : \mathbb{Z}_{2^q} \rightarrow \mathbb{Z}_{2^p}$ removes the $q - p$ LSBs.
- Security level is determined by n , q and $q - p$: noise $\sim \text{Uniform}[-2^{q-p}, 0)$

e.g. LightSaber: $n = 512$, $q - p = 3$, dPRF LaKey $n = 256$, $q - p = 4$.

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Hardness

Theory

- **LWE**: Solid theoretical foundations (e.g. Brakerski et al. 13).
- **LWR** is **as hard as** LWE (asymptotic reduction, underlying assumptions).

Practice

- Parameter selection driven by **best known attacks** (Lattice estimator, Albrecht et al.)

'The hardness of (ring or module) LWR can be analyzed as an LWE problem, since there is no known attacks that make use of the additional structure offered by these variants'.

SABER specifications

Open question: what does a deterministic error do to (practical) security?

Linearisation attack by Arora & Ge (2011)

Parameters: $n \in \mathbb{N}^*$, Noise in set E .

Any sample $(\mathbf{a}, s_{\mathbf{a}})$, yields the following equation over \mathbb{Z}_{2^q} in the unknowns $\mathbf{x} = (x_0, \dots, x_{n-1})$

$$\prod_{e \in E} \left(\sum_{i=0}^{n-1} \mathbf{a}_i \times x_i - e - s_{\mathbf{a}} \right) = 0.$$

Linearisation: $\binom{n+|E|}{|E|}$ in data, $\binom{n+|E|}{|E|}^\omega$ in time, ω linear algebra constant.

- **LWE:** Gaussian distribution: bounded noise for a well-chosen number of samples.
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Our observation: inapplicable for some parameter regimes **independently of the nr of samples**.

Our main result: in the case of LWR, one can do an attack that 1) works 2) better.



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$$\mathbb{Z}_{2^q} \simeq \{0, 1, 2, \dots, 2^q - 1\} \simeq \mathbb{F}_2^q$$

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- n unknowns \mathbf{x}_i in $\mathbb{Z}_{2^q} \simeq nq$ binary unknowns $(\mathbf{x}_{i,j})_{\substack{0 \leq i \leq n-1 \\ 0 \leq j \leq q-1}} \in \mathbb{F}_2^{nq}$.

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The LWR function is a $(2nq, p)$ -vectorial Boolean function:

$$F : \mathbb{F}_2^{nq} \times \mathbb{F}_2^{nq} \rightarrow \mathbb{F}_2^p \quad (\mathbf{a}, \mathbf{x}) \mapsto \left[\sum_{i=0}^{n-1} \mathbf{a}_i \times \mathbf{x}_i \right]_{2^p}$$

The **LWR problem** can be studied in a “symmetric” manner (\simeq weak-PRF).

Exponential notation

If $u_0, \dots, u_{n-1}, v_0, \dots, v_{n-1}$ are q -bit integers:

$$\mathbf{a}^u \mathbf{x}^v := \prod_i \prod_j a_{i,j}^{u_{i,j}} x_{i,j}^{v_{i,j}}$$

Consider $M = a_{0,0} a_{0,1} a_{1,0} a_{1,1} a_{1,2} x_{0,0} x_{1,0} x_{1,1}$.

We denote it by $M = a_0^{0b011} a_1^{0b111} x_0^{0b001} x_1^{0b011} = \mathbf{a}^{(3,7)} \mathbf{x}^{(1,3)}$.

Algebraic Normal Form

Algebraic Normal Form (ANF). Any Boolean function $f : \mathbb{F}_2^s \rightarrow \mathbb{F}_2$ admits a unique multivariate polynomial form:

$$\forall x \in \mathbb{F}_2^s, \quad f(x) = \sum_{u \in \mathbb{F}_2^s} \alpha_u(f) x^u.$$

Product of coordinates. For any $F : \mathbb{Z}_{2^q} \rightarrow \mathbb{Z}_{2^q}$, and any $m = \sum_i m_i 2^i$, recall:

$$x \mapsto (F(x))^m := \prod_i F_i(x)^{m_i}.$$

In the following, we study (products of) coordinates of the inner product:

$$F^{m,n} : (\mathbf{a}, \mathbf{x}) \mapsto (\langle \mathbf{a}, \mathbf{x} \rangle)^m.$$

$\mathbb{F}_2[\mathbf{a}, \mathbf{x}]$ or $\mathbb{F}_2[\mathbf{a}][\mathbf{x}]$?

$$F^{m,n} : (\mathbb{Z}_{2^q})^n \times (\mathbb{Z}_{2^q})^n \longrightarrow \mathbb{F}_2$$
$$(\mathbf{a}, \mathbf{x}) \longmapsto (\langle \mathbf{a}, \mathbf{x} \rangle)^m .$$

If $m = 2^{q-p}$: coordinate of index $q - p$
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$$F^{m,n} = \sum_{\mathbf{v}} \underbrace{\left(\sum_{\mathbf{u}} \alpha_{\mathbf{u}, \mathbf{v}} \mathbf{a}^{\mathbf{u}} \right)}_{\alpha_{\mathbf{v}}} \mathbf{x}^{\mathbf{v}}, \quad \alpha_{\mathbf{v}} \in \mathbb{F}_2[\mathbf{a}].$$

- Cost of linearisation $\leq \#\text{monomials}^{\omega} = |\text{Exp}_{\mathbf{x}}(F^{m,n})|^{\omega}$ with $\text{Exp}_{\mathbf{x}}(F^{m,n}) = \{\mathbf{v} \mid \alpha_{\mathbf{v}} \neq 0\}$.
- Linearisation is possible only if the ANF of each $\alpha_{\mathbf{v}}$ is known (We'll get back to it...)



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Set of exponents of $F^{m,n}$

Ordered integer partitions. Let $S_k^n = \{v \in \mathbb{N}^n : \sum_{i=0}^{n-1} v_i = k\}$ be the set of n -long vectors that sum to k .

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Theorem (Exponents of $F^{m,n}$).

$$\text{Exp}_x(F^{m,n}) \subset \bigcup_{k=1}^m S_k^n.$$

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- Proven using [Braeken & Semaev FSE05].

If $G(\mathbf{x}) = x_0 + x_1 + \dots + x_{n-1} \in \mathbb{Z}_{2^q}$, then $\text{Exp}_{\mathbf{x}}(G^m) = S_m^n$

- Related to the properties of **S-functions**.

[MouCanIndPre09]

An attack that works and costs the same (in fact, a little less)

- **Number of monomials.** $|\text{Exp}_x(F^{m,n})| \leq |\bigcup_{k=1}^m S_k^n| = \binom{n+m}{m}$.

Open question: we conjecture equality when m is a power of two.

- **Degree.** $\deg(F^{m,n}) \leq m$. Equality holds when $m \leq n$.

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- Recall over \mathbb{Z}_{2^q} (when applicable): $\binom{n+2^q-p}{2^q-p}$ monomials.

[Arora & Ge]

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- If we use the LSB of the sample ($m = 2^{q-p}$), **same number of monomials** (Surprising!).
- Working over \mathbb{F}_2 rather than \mathbb{Z}_{2^q} : we gain at least 2^{q-p} .
 - 1 it's a field.
 - 2 operations are **cheaper**

The general strategy

- 1 With $\binom{n+2^{q-p}}{2^{q-p}}$ samples, recover the $q - p$ LSBs of each x_i by linearisation.
- 2 Observe that each bit of $\langle \mathbf{a}, \mathbf{x} \rangle$ satisfies $b_j = \sum_{i=0}^{n-1} \mathbf{a}_{i,j} x_{i,j} + c_j$ where c_j is a carry.
- 3 Compute c_j from the known $\mathbf{a}_{i,k}$'s and recovered $x_{i,k}$'s, $k < j$.
- 4 Solve a linear system in $x_{i,j}$ only.
- 5 Repeat for increasing j .

The cost of steps 2 to 5 is negligible before the cost of step 1.



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This is *a priori* hard:

- Möbius transform is out of reach (the LUT cannot be fully computed/stored).
- Direct computations using recursive formulas seem hopeless.

Our results

The ANF can be 'understood' for **arbitrary large n** and for **m up to 16** by:

- storing the ANF for $n = m$.
- computing properties for arbitrary large n .

Additional improvement of the attack

Default linearisation: one auxiliary variable per monomial $\mathbf{x}^{\mathbf{v}}$ in the ANF of $F^{m,n}$.

Example.

$$F(\mathbf{a}, \mathbf{x}) = \alpha_{\mathbf{v}} \mathbf{x}^{\mathbf{v}} + \alpha_{\mathbf{v}'} \mathbf{x}^{\mathbf{v}'} + \alpha_{\mathbf{v}''} \mathbf{x}^{\mathbf{v}''} + \dots \quad \alpha_{\mathbf{v}}, \alpha_{\mathbf{v}'}, \alpha_{\mathbf{v}''} \in \mathbb{F}_2[\mathbf{a}]$$

- If $\alpha_{\mathbf{v}} = \alpha_{\mathbf{v}'} = \alpha_{\mathbf{v}''}$, then $\mathbf{x}^{\mathbf{v}}$, $\mathbf{x}^{\mathbf{v}'}$, $\mathbf{x}^{\mathbf{v}''}$ appear/vanish together for any value of \mathbf{a} .
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- Less auxiliary variables: improved linearisation (data and time).
- Ideally: compute the rank and a basis for $\{\alpha_{\mathbf{v}}, \mathbf{v} \in \text{Exp}(F^{m,n})\}$.

Work in progress

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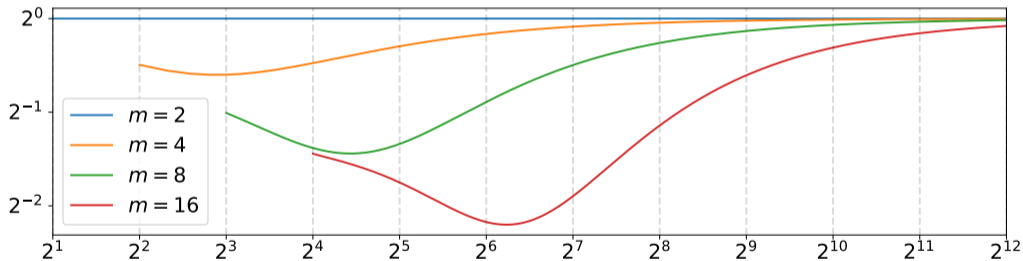
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- In practice, we compute the generating family $Q^{m,n}$ made of distinct $\alpha_{\mathbf{v}}$'s. $m \leq 16$.
- The average sparsity for both families (monomials, $Q^{m,n}$). $m \leq 8$.

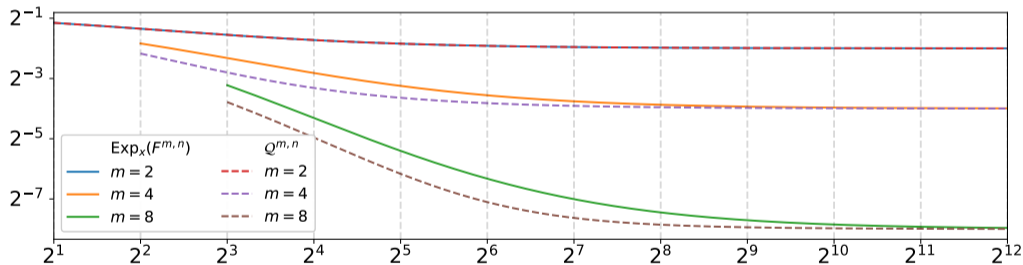
Our results: Upper bound on the rank

Ratio $|Q^{m,n}| / \binom{n+m}{m} - 1$ as a function of n , for $n \in \llbracket m, 4096 \rrbracket$.



Our results: Sparsity

Average fraction of terms in a random equation $F_a^{m,n}(x)$ as a function of n .



Improvements over Arora-Ge

Comparison to the linearisation attack by Arora & Ge using $\omega = 3$.

and cost of modular addition/multiplication $m = 2^{q-p}$.

$m = 2^{q-p}$	n	Arora-Ge	Our work (\leq rank only)	Our work (\leq rank and sparsity)
8	64	$2^{103.4}$	$2^{97.8}$	$2^{87.2}$
	128	$2^{126.3}$	$2^{121.8}$	$2^{110.8}$
	256	$2^{149.7}$	$2^{145.9}$	$2^{134.7}$
16	64	$2^{167.7}$	$2^{157.2}$	Non-available
	128	$2^{211.7}$	$2^{202.00}$	Non-available
	256	$2^{257.5}$	$2^{250.1}$	Non-available



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Overview

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The ANF (and additional properties) can be understood for *arbitrary large* n and for m up to 16.

NB: these m 's are used in practice e.g. $m = 8$ for SABER, $m = 16$ for LaKEY.

We take advantage of symmetries due to the *commutativity of modular addition*.

Observe that:

- $\langle (a_0, a_1), (x_0, x_1) \rangle = a_0x_0 + a_1x_1 = a_1x_1 + a_0x_0 = \langle (a_1, a_0), (x_1, x_0) \rangle$.

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Overview

Our result

The ANF (and additional properties) can be understood for **arbitrary large n** and for **m up to 16**.

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- More generally for any $n \leq n'$, $\langle \mathbf{a} || 0^{n'-n}, \mathbf{x} || 0^{n'-n} \rangle_{n'} = \langle \mathbf{a}, \mathbf{x} \rangle_n$.

Reduction to a system of representatives

- The group \mathfrak{S}_n acts on vectors of length n : $\sigma \cdot \mathbf{u} := (u_{\sigma^{-1}(0)}, \dots, u_{\sigma^{-1}(n-1)})$.
- $F(\mathbf{a}, \mathbf{x})$ is \mathfrak{S}_n -invariant if $F(\mathbf{a}, \mathbf{x}) = F(\sigma \cdot \mathbf{a}, \sigma \cdot \mathbf{x})$ for all $\sigma \in \mathfrak{S}_n$.

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Example. The function $F : (\mathbf{a}, \mathbf{x}) \mapsto \mathbf{a}^{(1,0)} \mathbf{x}^{(3,0)} + \mathbf{a}^{(0,1)} \mathbf{x}^{(0,3)}$ is \mathfrak{S}_2 -invariant (not symmetric!)
Indeed, $\mathfrak{S}_2 = \{\text{id}, (0\ 1)\}$ and $(0\ 1) \cdot F = F((0\ 1) \cdot \mathbf{a}, (0\ 1) \cdot \mathbf{x}) = F(\mathbf{a}, \mathbf{x})$.

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For any m, n , $F^{m,n}$ is \mathfrak{S}_n -invariant. \mathfrak{S}_n -invariance is equivalent to:

- $\forall \mathbf{u}, \mathbf{v}, \sigma, \quad \alpha_{(\sigma \cdot \mathbf{u}, \sigma \cdot \mathbf{v})} = \alpha_{(\mathbf{u}, \mathbf{v})} \quad \alpha_{\mathbf{u}, \mathbf{v}} \in \mathbb{F}_2$
- and to $\forall \mathbf{u}, \sigma, \quad \sigma \cdot \alpha_{\mathbf{v}} = \alpha_{\sigma^{-1} \cdot \mathbf{v}} \quad \alpha_{\mathbf{v}} \in \mathbb{F}_2[\mathbf{a}]$

Allows to represent the ANF in a compact way.

Scaling

How do we get results for arbitrary n ?

Example. Let's look at the \mathfrak{S}_2 -invariant function

$$F^{(2)}((a_0, a_1), (x_0, x_1)) = \mathbf{a}^{(1,0)} \mathbf{x}^{(3,0)} + \mathbf{a}^{(0,1)} \mathbf{x}^{(0,3)} .$$

then there exists a \approx unique \mathfrak{S}_3 -invariant function $F^{(3)}$ such that

$$((a_0, a_1), (x_0, x_1)) \mapsto F^{(3)}((a_0, a_1, 0), (x_0, x_1, 0))$$

is equal to $F^{(2)}$:

$$F^{(3)}((a_0, a_1, a_2), (x_0, x_1, x_2)) \mapsto \mathbf{a}^{(1,0,0)} \mathbf{x}^{(3,0,0)} + \mathbf{a}^{(0,1,0)} \mathbf{x}^{(0,3,0)} + \mathbf{a}^{(0,0,1)} \mathbf{x}^{(0,0,3)} .$$

(Properties of) $F^{m,n}$ can be derived from $F^{m,m}$.



Outline

- 1 Introduction: motivation and setting
- 2 A symmetric point of view
- 3 Results
- 4 Some intuition
- 5 A taste of the algorithms**

Group actions

Let \mathbb{G} be a group acting on a set E . Let $x \in E$.

- **Orbit.** $\text{Orb}(x) := \{\sigma \cdot x, \sigma \in \mathbb{G}\} \subset E$.
- **Stabilizer.** $\text{Stab}(x) := \{\sigma, \sigma \cdot x = x\} < \mathbb{G}$.

Example. Let $\mathbf{v} = (3, 1, 1)$. Then $\text{Orb}(\mathbf{v}) = \{(3, 1, 1), (1, 3, 1), (1, 1, 3)\}$ and $\text{Stab}(\mathbf{v}) = \{\text{id}, (1\ 2)\}$.

Important properties.

- The set of orbits $\{\text{Orb}(x), x \in E\}$ is a partition of E .
- It induces an equivalence relation: $x \sim x'$ if and only if $x' \in \text{Orb}(x)$.
- For any $x \in E$,

$$|\text{Orb}(x)| \cdot |\text{Stab}(x)| = |\mathbb{G}| .$$

Effective computation of the ANF

Recall: S_m^n be the set of n -long vectors that sum to m .

$$F^{m,n}(\mathbf{a}, \mathbf{x}) = \langle \mathbf{a}, \mathbf{x} \rangle^m = \left(\sum_i a_i \times x_i \right)^m \stackrel{[BraSem05]}{=} \sum_{\mathbf{c} \in S_m^n} \prod_i (a_i \times x_i)^{c_i}.$$

Let \mathcal{C}_m^n be a system of representatives of S_m^n :

(e.g. **unordered** partitions)

$$F^{m,n} = \dots = \underbrace{\sum_{\mathbf{c} \in \mathcal{C}_m^n} \sum_{\mathbf{c}' \in \text{Orb}(\mathbf{c})}}_{G_{\mathbf{c}}} \overbrace{\prod_i (a_i \times x_i)^{c'_i}}^{H_{\mathbf{c}'}}$$

We observe:

- $G_{\mathbf{c}}$ is \mathfrak{S}_n -invariant.
- $H_{\sigma \cdot \mathbf{c}} = \sigma^{-1} \cdot H_{\mathbf{c}}$.

Effective computation of the ANF

Theorem. Let $\mathbf{c}, \mathbf{u}, \mathbf{v} \in \mathbb{Z}_{2^q}^n$. Define $E_{\mathbf{c}}(\mathbf{u}, \mathbf{v}) := \text{Exp}(H_{\mathbf{c}}) \cap \text{Orb}(\mathbf{u}, \mathbf{v})$.

$$\alpha_{(\mathbf{u}, \mathbf{v})}(G_{\mathbf{c}}) = \frac{|E_{\mathbf{c}}(\mathbf{u}, \mathbf{v})| n!}{|\text{Stab}(\mathbf{c})| |\text{Orb}(\mathbf{u}, \mathbf{v})|} \pmod{2}.$$

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We provide algorithms (and implementations) to:

- (i) compute the ANF of $H_{\mathbf{c}^*}$ for $\mathbf{c}^* \in \mathcal{C}_m^n$.
- (ii) compute a SoR of the ANF of $G_{\mathbf{c}}$ (Theorem).

$$\{(\mathbf{u}, \mathbf{v})^*, \alpha_{(\mathbf{u}, \mathbf{v})^*}(G_{\mathbf{c}}) = 1\}.$$

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(non-trivial from (iii)).

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- (v) compute $\{\alpha_{\mathbf{v}} : \mathbf{v} \in \text{Exp}_x(F^{m,n})\} / \sim \subset \mathbb{F}_2[\mathbf{a}]$.

$$\begin{aligned} & \{(\mathbf{u}, \mathbf{v})^*, \alpha_{(\mathbf{u}, \mathbf{v})^*}(G_{\mathbf{c}}) = 1\}. \\ & \{(\mathbf{u}, \mathbf{v})^*, \alpha_{(\mathbf{u}, \mathbf{v})^*}(F^{m,n}) = 1\}. \\ & \text{(non-trivial from (iii)).} \\ & \text{(even less trivial).} \end{aligned}$$

Ex. of scaling: number of monomials

- Our algorithm (iv) returns $\{(\mathbf{v}^*, \alpha_{\mathbf{v}^*} \neq 0)\}$.

$$\text{Nr of monomials} = |\text{Exp}_{\mathbf{x}}(F^{m,n})| = \sum_{\mathbf{v}^*, \alpha_{\mathbf{v}^*} \neq 0} \text{Orb}(\alpha_{\mathbf{v}^*}) \binom{n}{|\text{Supp}(\mathbf{v}^*)|}$$

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In practice, when m is a power of two, this is **always** equal to the upper bound $\binom{n+m}{m} - 1$.

- In general, scaling is not the hard part (e.g. similar formula for scaling the average sparsity).

Conclusion

Results

- Deterministic noise impacts attacks.
- Generic improvement (and “correction”) of Arora & Ge:
 - same number of monomials + working over \mathbb{F}_2 .
 - computation of additional parameters.

Many open questions

- Understanding the ANF even better (e.g. rank).
- Help us solve this system (with less data)!
 - E.g. super structured ANF guides guess-and-solve strategies.
- More applications of group actions.